

The matrix function e^{tA+B} is representable as the Laplace transform of a matrix measure.

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Abstract. Given a pair A, B of matrices of size $n \times n$, we consider the matrix function e^{tA+B} of the variable $t \in \mathbb{C}$. If the matrix A is Hermitian, the matrix function e^{tA+B} is representable as the bilateral Laplace transform of a matrix-valued measure $M(d\lambda)$ compactly supported on the real axis:

$$e^{tA+B} = \int e^{t\lambda} M(d\lambda).$$

The values of the measure $M(d\lambda)$ are matrices of size $n \times n$, the support of this measure is contained in the convex hull of the spectrum of A . If the matrix B is also Hermitian, then the values of the measure $M(d\lambda)$ are Hermitian matrices. The measure $M(d\lambda)$ is not necessarily non-negative.

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Notation

\mathbb{C} is the set of complex numbers.

\mathbb{R} is the set of real numbers.

\mathbb{R}^+ is the set of non-negative real numbers.

\mathbb{N} is the set of natural numbers.

\mathfrak{M}_n is the set of matrices of size $n \times n$ with entries belonging to \mathbb{C} .

\mathfrak{M}_n^+ is the set of matrices of size $n \times n$ with entries belonging to \mathbb{R}^+ .

\mathfrak{H}_n is the set of Hermitian matrices of size $n \times n$.

\mathfrak{D}_n is the set of diagonal matrices of size $n \times n$.

I_n is the identity matrix of size $n \times n$.

We provide the set \mathfrak{M}_n with the usual algebraic operations - the matrix addition and the matrix multiplication.

1. The goal of the present paper.

Let $A \in \mathfrak{H}_n$ and $B \in \mathfrak{M}_n$. In the present paper we consider the matrix function $L(t) = e^{tA+B}$ of the complex variable t . We show that this function is representable as the bilateral Laplace transform of some matrix valued measure $M(d\lambda)$:

$$e^{tA+B} = \int e^{t\lambda} M(d\lambda), \quad t \in \mathbb{C}, \quad (1.1)$$

the values of the measure M belong to the set \mathfrak{M}_n .

Our considerations are based on the functional calculus for the matrix A . We relate the following objects with the matrix A :

1. The spectrum $\sigma(A)$ of the matrix A , that is the set $\{\lambda_1, \dots, \lambda_l\}$ of all its eigenvalues taken without multiplicities, i.e. $\lambda_p \neq \lambda_q, \forall p \neq q, 1 \leq p, q \leq l$. Since $A \in \mathfrak{H}_n$, $\sigma(A) \subset \mathbb{R}$. (The number l is the cardinality of the set $\sigma(A)$, $l \leq n$. If the spectrum $\sigma(A)$ is simple, then $l = n$.)
2. The set $\{E_{\lambda_1}, \dots, E_{\lambda_l}\}$ of spectral projectors of the matrix A :

$$AE_{\lambda_j} = \lambda_j E_{\lambda_j}, \quad 1 \leq j \leq l, \quad (1.2)$$

$$E_{\lambda_1} + \dots + E_{\lambda_l} = I_n. \quad (1.3)$$

If $f(\lambda)$ is a function defined on the spectrum $\sigma(A)$, then

$$f(A) = \sum_{1 \leq j \leq l} f(\lambda_j) E_{\lambda_j}. \quad (1.4)$$

In particular,

$$e^{tA} = \sum_{1 \leq j \leq l} e^{t\lambda_j} E_{\lambda_j}. \quad (1.5)$$

If the matrices A and B commute, that is if

$$AB = BA, \quad (1.6)$$

then

$$e^{tA+B} = e^{tA} \cdot e^{tB}. \quad (1.7)$$

From (1.5) and (1.7) it follows that under the condition (1.6) the equality

$$e^{tA+B} = \sum_{1 \leq j \leq l} e^{t\lambda_j} M(\{\lambda_j\}) \quad (1.8)$$

holds, where

$$M(\{\lambda_j\}) = E_{\lambda_j} e^{tB} E_{\lambda_j}. \quad (1.9)$$

The equality (1.8) can be interpreted as the representation of the matrix function e^{tA+B} in the form of the bilateral Laplace transform (1.1) of a very special matrix valued measure M . This measure M is discrete and is supported on the spectrum $\sigma(A)$ of the matrix A . The point $\{\lambda_j\} \in \sigma(A)$ carries the "atom" $M(\{\lambda_j\})$.

The goal of the present paper is to obtain the representation of the matrix function e^{tA+B} in the form (1.1) without assuming that the matrices A and B commute.

The representation of the form (1.1) was suggested by the following

Theorem. (H.Stahl). *Let matrices A and B be given, $A \in \mathfrak{H}_n$, $B \in \mathfrak{H}_n$. Let λ_{\min} and λ_{\max} be the smallest and the largest eigenvalues of the matrix A . Then the function $\text{trace } e^{At+B}$ is representable in the form*

$$\text{trace } e^{tA+B} = \int_{[\lambda_{\min}, \lambda_{\max}]} e^{t\lambda} \mu(d\lambda), \quad (1.10)$$

where $\mu(d\lambda)$ is a non-negative Borel measure.

The first arXiv version of Stahl's Theorem appeared in [1], the latest arXiv version - in [2], the journal publication - in [3]. The proof of Stahl is based on ingenious considerations related to Riemann surfaces of algebraic functions. In [4],[5] a simplified version of Stahl proof is presented. The proof, presented in [4],[5], preserves all the main ideas of Stahl; the simplification consists in technical details. In the paper [6] a proof of Stahl's Theorem for the special case $\text{rank } A = 1$ is presented. This proof is based on an elementary argument which does not require complex analysis.

The main result of the present paper is Theorem 7.1. Stahl's Theorem does not follow from our Theorem 7.1. If $A \in \mathfrak{H}_n$ and $B \in \mathfrak{H}_n$, then the measure $M(d\lambda)$ in (1.1) is \mathfrak{H}_n -valued but not necessarily is non-negative. An appropriate example is given in Section 8.

2. The approximant $L_N(t)$.

If the matrices A and B do not commute, then the equality (1.7) breaks down. However the Lie product formula, which is a kind of surrogate for the formula (1.7), holds regardless of the condition (1.6).

Lie Product Formula. *Let $X \in \mathfrak{M}_n$ and $Y \in \mathfrak{M}_n$. Then*

$$e^{X+Y} = \lim_{N \rightarrow \infty} (e^{X/N} e^{Y/N})^N. \quad (2.1)$$

Versions of proof of the Lie Product Formula¹ can be found in [7], [8, Section 6.5], [9, Theorem 2.10], [10, Section 2.12, Corollary 2.12.5].

Lemma 2.1. *Let $A \in \mathfrak{M}_n$, $B \in \mathfrak{M}_n$ and $t \in \mathbb{C}$. Then*

$$e^{tA+B} = \lim_{N \rightarrow \infty} (e^{tA/N} e^{B/N})^N. \quad (2.2)$$

Proof. The formula (2.2) is a special case of the formula (2.1) corresponding to the choice $X = tA$, $Y = B$. \square

¹ It should be mention that the Lie Product Formula can be extended to certain unbounded linear operators X and Y acting in a Hilbert space. First such extension was done by Trotter, [11]. A version of the Trotter-Lie product formula was obtained by T.Kato, [12]. We refer also to the book of B.Simon, [13]. See Theorems 1.1 and 1.2 there. In [13], the Lie-Trotter formula is used for the path integral formulation of quantum mechanics.

Definition 2.2. The expression

$$L_N(t) = \left(e^{tA/N} e^{B/N} \right)^N \quad (2.3)$$

which appears on the right hand side of (2.3) is said to be the *N-approximant for the matrix function* $L(t) = e^{tA+B}$.

Assuming that $A \in \mathfrak{H}_n$, we express the matrix function $e^{tA/N}$ in terms of the spectrum $\sigma(A)$ of the matrix A and its spectral projectors:

$$e^{tA/N} = \sum_{1 \leq j \leq l} e^{t \frac{\lambda_j}{N}} E_{\lambda_j}. \quad (2.4)$$

Substituting (2.4) into (2.3), we represent the approximant $L_N(t)$ as a multiple sum which contains l^N summands:

$$L_N(t) = \sum_{k_1, \dots, k_N} \exp \left\{ t \frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N} \right\} M_{k_1, \dots, k_N}. \quad (2.5)$$

In (2.5), the summation is extended over all integers k_1, \dots, k_N such that $1 \leq k_p \leq l$, $p = 1, 2, \dots, N$. The matrix M_{k_1, \dots, k_N} is the product

$$M_{k_1, \dots, k_N} = E_{\lambda_{k_1}} e^{B/N} E_{\lambda_{k_2}} e^{B/N} \dots E_{\lambda_{k_N}} e^{B/N}. \quad (2.6)$$

Let us consider the numbers $\frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N}$ which appear in the exponents of the exponentials in (2.5).

Lemma 2.3. *Given integers k_1, \dots, k_N satisfying the conditions $1 \leq k_p \leq l$, $p = 1, 2, \dots, N$, then*

$$\frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N} = \frac{n_1}{N} \lambda_1 + \frac{n_2}{N} \lambda_2 + \dots + \frac{n_l}{N} \lambda_l, \quad (2.7)$$

where

$$n_j(k_1, \dots, k_N) = \#\{p : 1 \leq p \leq N, k_p = j\}, \quad 1 \leq j \leq l. \quad (2.8)$$

The numbers

$$\xi_j = \frac{n_j}{N}, \quad j = 1, 2, \dots, l, \quad (2.9)$$

where the n_j are defined by (2.8), satisfy the conditions

$$\xi_j \geq 0, \quad j = 1, 2, \dots, l, \quad \sum_{1 \leq j \leq l} \xi_j = 1. \quad (2.10)$$

Proof. The lemma is evident. \square

The linear combination $\xi_1 \lambda_1 + \xi_2 \lambda_2 + \dots + \xi_l \lambda_l$ which appears on the right hand side of (2.7) is a *convex* linear combination of numbers $\lambda_1, \lambda_2, \dots, \lambda_l$. However this linear combination is a very special convex linear combination. Its coefficients ξ_1, \dots, ξ_l are numbers of the form $\xi_j = \frac{n_j}{N}$, where n_j are non-negative integers.

Definition 2.4. Let $\lambda_1, \dots, \lambda_l$ be real numbers, N be a positive integer. The N -convex hull of the set $\{\lambda_1, \dots, \lambda_l\}$ is the set $\xi_1\lambda_1 + \xi_2\lambda_2 + \dots + \xi_l\lambda_l$ of all convex linear combinations with coefficients of the form $\xi_j = \frac{n_j}{N}$, where n_j are non-negative integers. (Since the considered linear combinations are convex, the equality $n_1 + n_2 + \dots + n_l = N$ must hold.)

In what follows, the numbers $\lambda_1, \dots, \lambda_l$ form the spectrum $\sigma(A)$ of the matrix A . The N -convex hull of the spectrum $\sigma(A)$ is denoted by $ch_N(\sigma(A))$. The convex hull of the spectrum $\sigma(A)$ is denoted by $ch(\sigma(A))$.

Remark 2.5. It is clear that the convex hull $ch(\sigma(A))$ is the closed interval $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min} = \min_{1 \leq j \leq l} \lambda_j$, $\lambda_{\max} = \max_{1 \leq j \leq l} \lambda_j$. It is also clear, that

$$ch_N(\sigma(A)) \subset ch(\sigma(A)), \quad \forall N. \quad (2.11)$$

The union $\bigcup_N ch_N(\sigma(A))$ of the sets $ch_N(\sigma(A))$ is dense in the set $ch(\sigma(A))$.

The numbers $\frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N}$ which appear in the exponents of the exponentials in (2.5) belong to the set $ch_N(\sigma(A))$. Collecting similar terms, we rewrite (2.5) in the form

$$L_N(t) = \sum_{\lambda \in ch_N(\sigma(A))} e^{t\lambda} M_N(\{\lambda\}), \quad (2.12)$$

where

$$M_N(\{\lambda\}) = \sum_{k_1, \dots, k_N} M_{k_1, \dots, k_N}, \quad (2.13)$$

the matrices M_{k_1, \dots, k_N} are defined by (2.6). For each $\lambda \in ch_N(\sigma(A))$, the sum in (2.13) is extended over all those k_1, \dots, k_N for which $\frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N} = \lambda$.

We interpret the equality (2.12) as the representation of the approximant $L_N(t)$ in the form of the bilateral Laplace transform of a matrix valued measure $M_N(d\lambda)$:

$$L_N(t) = \int_{\lambda \in ch_N(\sigma(A))} e^{t\lambda} M_N(d\lambda). \quad (2.14)$$

The measure $M_N(d\lambda)$ is discrete and is supported on the finite set $ch_N(\sigma(A))$. The point $\{\lambda\} \in ch_N(\sigma(A))$ carries the "atom" $M_N(\{\lambda\})$.

According to (2.2)

$$e^{tA+B} = \lim_{N \rightarrow \infty} \int e^{t\lambda} M_N(d\lambda), \quad \forall t \in \mathbb{C}. \quad (2.15)$$

3. The norm in the set \mathfrak{M}_n .

We have to prove that the sequence $\{M_N(d\lambda)\}_{1 \leq N < \infty}$ of matrix measures is weakly convergent. To prove this, we have to bound the total variations of these measures from above.

To express such bound, we have to provide the set \mathfrak{M}_n with some norm. We provide the set \mathfrak{M}_n with the usual *operator norm*. Let $S = (s_{pq})_1^n \in \mathfrak{M}_n$. The norm $\|S\|$ is defined as follows:

$$\|S\| \stackrel{\text{def}}{=} \max_{\xi, \eta} \frac{\left| \sum_{1 \leq p, q \leq n} s_{pq} \xi_q \eta_p \right|}{\sqrt{|\xi_1|^2 + \dots + |\xi_n|^2} \sqrt{|\eta_1|^2 + \dots + |\eta_n|^2}}, \quad (3.1)$$

where max is taken over all complex numbers ξ_1, \dots, ξ_n and η_1, \dots, η_n

Lemma 3.1. *Let $S = (s_{pq})_1^n \in \mathfrak{M}_n$. Then the inequality*

$$\|S\| \leq \sum_{1 \leq p, q \leq n} |s_{pq}| \quad (3.2)$$

holds.

Proof. The inequality (3.2) is a direct consequence of the inequality

$$\left| \sum_{1 \leq p, q \leq n} s_{pq} \xi_q \eta_p \right| \leq \left(\sum_{1 \leq p, q \leq n} |s_{pq}| \right) \cdot \max_{1 \leq p, q \leq n} |\xi_q \eta_p|$$

and of the inequality

$$\max_{1 \leq p, q \leq n} |\xi_q \eta_p| \leq \sqrt{|\xi_1|^2 + \dots + |\xi_n|^2} \sqrt{|\eta_1|^2 + \dots + |\eta_n|^2}. \quad \square$$

Lemma 3.2. *Let $S = (s_{pq})_1^n \in \mathfrak{M}_n^+$. Then the inequality*

$$\sum_{1 \leq p, q \leq n} s_{pq} \leq n \cdot \|S\| \quad (3.3)$$

holds.

Proof. The ratio $\frac{\sum_{1 \leq p, q \leq n} s_{pq}}{n}$ can be considered as the ratio

$$\frac{\sum_{1 \leq p, q \leq n} s_{pq} \xi_q \eta_p}{\sqrt{|\xi_1|^2 + \dots + |\xi_n|^2} \sqrt{|\eta_1|^2 + \dots + |\eta_n|^2}}$$

with $\xi_1 = 1, \dots, \xi_n = 1$ and $\eta_1 = 1, \dots, \eta_n = 1$. \square

The inequality expressed by following Lemma can be considered as an *inverse triangle inequality*. It holds for matrices with *non-negative* entries. The total number of summands can be arbitrary large.

Lemma 3.3. *Let $S_r \in \mathfrak{M}_n^+$, $r = 1, 2, \dots, m$. Then the inequality*

$$\sum_{1 \leq r \leq m} \|S_r\| \leq n \cdot \left\| \sum_{1 \leq r \leq m} S_r \right\| \quad (3.4)$$

holds.

Proof. Lemma 3.3 is a direct consequence of Lemmas 3.1 and 3.2. \square

The following technical result is used later in Section 6.

Lemma 3.4. *Let the following objects be given:*

1. Matrices $F_j \in \mathfrak{M}_n^+$, $j = 1, \dots, l$, which satisfy the condition

$$\sum_{1 \leq j \leq l} F_j = I_n; \quad (3.5)$$

2. A matrix $R \in \mathfrak{M}_n^+$ and a number $N \in \mathbb{N}$.

Then the inequality

$$\sum_{k_1, k_2, \dots, k_N} \|F_{k_1} e^{R/N} F_{k_2} e^{R/N} \dots F_{k_N} e^{R/N}\| \leq n \cdot \|e^R\| \quad (3.6)$$

holds. The summation in (3.6) is extended over all integers k_1, k_2, \dots, k_N which satisfy the conditions² $1 \leq k_1 \leq l, 1 \leq k_2 \leq l, \dots, 1 \leq k_N \leq l$.

Proof. According to Lemma 3.3, the inequality

$$\begin{aligned} \sum_{k_1, k_2, \dots, k_N} \|F_{k_1} e^{R/N} F_{k_2} e^{R/N} \dots F_{k_N} e^{R/N}\| \leq \\ n \cdot \left\| \sum_{k_1, k_2, \dots, k_N} F_{k_1} e^{R/N} F_{k_2} e^{R/N} \dots F_{k_N} e^{R/N} \right\| \end{aligned}$$

holds. From the condition (3.5) it follows that

$$\sum_{k_1, k_2, \dots, k_N} F_{k_1} e^{R/N} F_{k_2} e^{R/N} \dots F_{k_N} e^{R/N} = e^{R/N} \cdot e^{R/N} \cdot \dots \cdot e^{R/N} = e^R.$$

□

Remark 3.5. If $F_j \in \mathfrak{M}_n^+$, $\forall j = 1, \dots, l$, and the equality (3.5) holds, then $F_j \in \mathfrak{D}_n$, $\forall j = 1, \dots, l$. The diagonal entries of each of the matrices F_j belong to the interval $[0, 1]$.

4. An expression for the total variation of the measure $M_N(d\lambda)$.

Since the measure $M_N(d\lambda)$ is discrete and its support is a finite set $ch_N(\sigma(A))$, the total variation of this measure is expressed by the sum

$$\sum_{\lambda \in ch_N(\sigma(A))} \|M_N(\{\lambda\})\|.$$

To prove that the family of measures $\{M_N(d\lambda)\}_{1 \leq N < \infty}$ is weakly convergent, we have to obtain an estimate of the form

$$\sum_{\lambda \in ch_N(\sigma(A))} \|M_N(\{\lambda\})\| \leq C, \quad \forall N, \quad (4.1)$$

where $C < \infty$ does not depend on N .

² So the sum in (3.6) contains l^N summands.

Lemma 4.1. *Let $M_N(\{\lambda\})$, $\lambda \in ch_N(\sigma(A))$, be the matrices which appear in the representation (2.12)-(2.13) of the N -approximant $L_N(t)$. Then the inequality*

$$\sum_{\lambda \in ch_N(\sigma(A))} \|M_N(\{\lambda\})\| \leq \sum_{k_1, \dots, k_N} \|M_{k_1, \dots, k_N}\|, \quad (4.2)$$

holds, where M_{k_1, \dots, k_N} are the same as in (2.6). On the right hand side of (4.2), the summation is extended over all integers k_1, \dots, k_N satisfying the conditions $1 \leq k_1 \leq l, \dots, 1 \leq k_N \leq l$.

Proof. Applying the triangle inequality to (2.13), we obtain the inequality

$$\|M_N(\{\lambda\})\| \leq \sum_{k_1, \dots, k_N} \|M_{k_1, \dots, k_N}\|, \quad \forall \lambda \in Nch(\sigma(A)). \quad (4.3)$$

On the right hand side of (4.3), the summation is extended over all those integers k_1, \dots, k_N for which $\frac{\lambda_{k_1} + \dots + \lambda_{k_N}}{N} = \lambda$. Adding the inequalities (4.3) over all $\lambda \in ch_N(\sigma(A))$, we come to the inequality

$$\sum_{\lambda \in ch_N(\sigma(A))} \|M_N(\{\lambda\})\| \leq \sum_{\lambda \in ch_N(\sigma(A))} \left(\sum_{k_1, \dots, k_N} \|M_{k_1, \dots, k_N}\| \right). \quad (4.4)$$

Regrouping summands in the right hand side of (4.4), we come to the inequality (4.2). \square

5. The subordination relation.

Definition 5.1. Let $M = (m_{pq})_1^n \in \mathfrak{M}_n$, $S = (s_{pq})_1^n \in \mathfrak{M}_n^+$. We say that *the matrix M is subordinated to the matrix S* and use the notation $M \preceq S$ for the subordination relation if the inequalities

$$|m_{pq}| \leq s_{pq}, \quad 1 \leq p, q \leq n, \quad (5.1)$$

hold for the entries m_{pq} , s_{pq} of the matrices M, S , respectively.

Lemma 5.2. *We assume that $M \in \mathfrak{M}_n, S \in \mathfrak{M}_n^+$, and $M \preceq S$. Then*

$$\|M\| \leq \|S\|. \quad (5.2)$$

Proof. Let m_{pq}, s_{pq} be the entries of the matrices M and S , and $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ be arbitrary complex numbers. Then the inequality

$$\begin{aligned} \left| \sum_{1 \leq p, q \leq n} m_{pq} \xi_p \eta_q \right| &\leq \sum_{1 \leq p, q \leq n} |m_{pq}| |\xi_p| |\eta_q| \leq \sum_{1 \leq p, q \leq n} s_{pq} |\xi_p| |\eta_q| \\ &\leq \|S\| \sqrt{|\xi_1|^2 + \dots + |\xi_n|^2} \sqrt{|\eta_1|^2 + \dots + |\eta_n|^2}. \end{aligned}$$

holds. \square

Definition 5.3. Given a matrix $B \in \mathfrak{M}_n$, we associate the matrix $R(B)$ with B . By definition,

$$R(B) = (r_{pq})_1^n, \quad r_{pq} = \|B\|, \quad 1 \leq p, q \leq n, \quad (5.3)$$

Lemma 5.4. *The matrix B is subordinated to the matrix $R(B)$.*

Proof. The entry b_{pq} of the matrix B satisfies the inequality $|b_{pq}| \leq \|B\| = r_{pq}$, $1 \leq p, q \leq n$. \square

Lemma 5.5.

1. The matrix $R(B)$ is a Hermitian matrix of rank one.
2. The norms of the matrix $R(B)$ and its exponential $e^{R(B)}$ are

$$\|R(B)\| = n\|B\|, \quad \|e^{R(B)}\| = e^{n\|B\|}. \quad (5.4)$$

Proof. The only non-zero eigenvalue of the matrix $R(B)$ is the number $n\|B\|$. \square

Lemma 5.6. Let $\Psi_k \in \mathfrak{M}_n$, $\Phi_k \in \mathfrak{M}_n^+$, $k = 1, \dots, m$. Assume that for each $k = 1, \dots, m$, the matrix Ψ_k is subordinated to the matrix Φ_k :

$$\Psi_k \preceq \Phi_k, \quad k = 1, \dots, m.$$

Then the subordination relations

$$\begin{aligned} \Psi_1 + \Psi_2 + \dots + \Psi_m &\preceq \Phi_1 + \Phi_2 + \dots + \Phi_m, \\ \Psi_1 \cdot \Psi_2 \cdot \dots \cdot \Psi_m &\preceq \Phi_1 \cdot \Phi_2 \cdot \dots \cdot \Phi_m \end{aligned}$$

hold for the sum and the product of these matrices.

Proof. The assertion of Lemma is a direct consequence of the definition of matrix addition and multiplication and of elementary properties of numerical inequalities. \square

Lemma 5.7. Let $X \in \mathfrak{M}_n^+$. Then $e^X \in \mathfrak{M}_n^+$. If $Y \in \mathfrak{M}_n$, $Y \preceq X$, then $e^Y \preceq e^X$.

Proof. According Lemma 5.6, the subordination relations $\frac{1}{m!}Y^m \preceq \frac{1}{m!}X^m$ hold for every $m = 0, 1, 2, \dots$. Using Lemma 5.6 once more, we conclude that $\sum_{0 \leq m < \infty} \frac{1}{m!}Y^m \preceq \sum_{0 \leq m < \infty} \frac{1}{m!}X^m$. \square

6. A bound for the total variation of the measure $M_N(d\lambda)$.

Lemma 6.1. Let $A \in \mathfrak{H}_n$, $B \in \mathfrak{M}_n$, $N \in \mathbb{N}$, and let the matrices M_{k_1, \dots, k_N} be defined according to (2.6).

Then the inequality

$$\sum_{1 \leq k_1, \dots, k_N \leq l} \|M_{k_1, \dots, k_N}\| \leq ne^{n\|B\|} \quad (6.1)$$

holds.

Proof.

1. We impose the additional condition: the matrix A is diagonal. So

$$A \in \mathfrak{H}_n \cap \mathfrak{D}_n. \quad (6.2)$$

Then all spectral projectors E_{λ_j} are diagonal matrices. Hence $E_{\lambda_j} \in \mathfrak{M}_n^+$, i.e.

$$E_{\lambda_j} \preceq E_{\lambda_j}, \quad j = 1, \dots, l. \quad (6.3)$$

Let the matrix $R(B)$ be defined according to Definition 5.3. By Lemma 5.4, $B \preceq R(B)$. By Lemma 5.7,

$$e^{B/N} \preceq e^{R(B)/N}. \quad (6.4)$$

By Lemma 5.6, the subordination relation

$$M_{k_1 \dots k_N} \preceq E_{\lambda_{k_1}} e^{R(B)/N} E_{\lambda_{k_2}} e^{R(B)/N} \dots E_{\lambda_{k_N}} e^{R(B)/N}$$

is satisfied for every k_1, \dots, k_N . By Lemma 5.2, the inequality

$$\|M_{k_1 \dots k_N}\| \leq \|E_{\lambda_{k_1}} e^{R(B)/N} E_{\lambda_{k_2}} e^{R(B)/N} \dots E_{\lambda_{k_N}} e^{R(B)/N}\|$$

holds. Adding the above inequalities, we obtain the inequality

$$\sum_{k_1, \dots, k_N} \|M_{k_1, \dots, k_N}\| \leq \sum_{k_1, \dots, k_N} \|E_{\lambda_{k_1}} e^{R(B)/N} E_{\lambda_{k_2}} e^{R(B)/N} \dots E_{\lambda_{k_N}} e^{R(B)/N}\| \quad (6.5)$$

To estimate the sum in the right hand side of (6.5), we apply Lemma 3.4 with $F_j = E_{\lambda_j}$, $R = R(B)$ and obtain the inequality

$$\sum_{k_1, \dots, k_N} \|E_{\lambda_{k_1}} e^{R(B)/N} E_{\lambda_{k_2}} e^{R(B)/N} \dots E_{\lambda_{k_N}} e^{R(B)/N}\| \leq n \|e^{R(B)}\|. \quad (6.6)$$

Now we refer to Lemma 5.5. The inequality (6.1) is a consequence of (6.5), (6.6) and (5.4).

2. The inequality (6.1) is proved under the extra assumption that the matrix A is diagonal. Now we get rid of the extra assumption, that the matrix A is diagonal.

Let A be an arbitrary matrix from \mathfrak{H}_n . There exists a unitary matrix U such that the matrix

$$A^d = UAU^* \quad (6.7)$$

is diagonal. Of course $A^d \in \mathfrak{H}_n$. Then we define the matrices

$$B^d = UBU^*, \quad E_{\lambda_j}^d = UE_{\lambda_j}U^*, \quad M_{k_1, \dots, k_N}^d = UM_{k_1, \dots, k_N}U^*. \quad (6.8)$$

The matrices $E_{\lambda_j}^d$ are the spectral projectors of the matrix A^d . The matrices M_{k_1, \dots, k_N}^d can be represented in the form

$$M_{k_1, \dots, k_N}^d = E_{\lambda_{k_1}}^d e^{B^d/N} E_{\lambda_{k_2}}^d e^{B^d/N} \dots E_{\lambda_{k_N}}^d e^{B^d/N} \quad (6.9)$$

Since the matrix A^d is diagonal, the inequality

$$\sum_{k_1, \dots, k_N} \|M_{k_1, \dots, k_N}^d\| \leq ne^{n\|B^d\|} \quad (6.10)$$

holds. It remains to note that

$$\|M_{k_1, \dots, k_N}^d\| = \|M_{k_1, \dots, k_N}\|, \quad \|B^d\| = \|B\|. \quad \square$$

Lemma 6.2. *Given the matrices $A \in \mathfrak{H}_n$, $B \in \mathfrak{M}_n$, let $M_N(d\lambda)$ be the matrix valued measure which appears in the representation (2.14) of the N -approximant $L_N(t)$ of the matrix function e^{tA+B} .*

The total variation of the measure $M_N(d\lambda)$ admits the bound

$$\sum_{\lambda \in ch_N(\sigma(A))} \|M_N(\{\lambda\})\| \leq ne^{n\|B\|}. \quad (6.11)$$

Proof. We combine the inequalities (4.2) and (6.1). □

7. The representation of the matrix function e^{tA+B} in the form of the Laplace transform of a matrix valued measure.

Theorem 7.1. *Let matrices A and B be given, $A \in \mathfrak{H}_n$, $B \in \mathfrak{M}_n$. Let λ_{\min} and λ_{\max} be the smallest and the largest eigenvalues of the matrix A , \mathfrak{B} be the class of Borel sets of the closed interval $[\lambda_{\min}, \lambda_{\max}]$.*

Then there exists a set function $M : \mathfrak{B} \rightarrow \mathfrak{M}_n$ such that:

1. *M is countably additive and regular on \mathfrak{B} , i.e. M is a regular Borel \mathfrak{M}_n -valued measure on $[\lambda_{\min}, \lambda_{\max}]$.*
2. *The equality*

$$e^{tA+B} = \int_{[\lambda_{\min}, \lambda_{\max}]} e^{t\lambda} M(d\lambda), \quad \forall t \in \mathbb{C}, \quad (7.1)$$

holds.

3. *If $B \in \mathfrak{H}_n$, then the measure M is \mathfrak{H}_n -valued, i.e. $M : \mathfrak{B} \rightarrow \mathfrak{H}_n$.*

Proof. Let us consider the Banach space $C([\lambda_{\min}, \lambda_{\max}])$ of \mathbb{C} -valued continuous functions on the interval $[\lambda_{\min}, \lambda_{\max}]$ equipped with the standard norm

$$\|x(\lambda)\| = \max_{\lambda \in [\lambda_{\min}, \lambda_{\max}]} |x(\lambda)|, \quad x(\lambda) \in C([\lambda_{\min}, \lambda_{\max}]).$$

The support of each of the measures $M_N(d\lambda)$ is contained in the closed interval $[\lambda_{\min}, \lambda_{\max}]$. (See Remark 2.5.) The total variation of the measures $M_N(d\lambda)$ is bounded from above by some finite value which does not depend on N . (See Lemma 6.2.) According to (2.15), for each $t \in \mathbb{R}$ there exists the limit

$$\lim_{N \rightarrow \infty} \int_{[\lambda_{\min}, \lambda_{\max}]} e^{t\lambda} M_N(d\lambda).$$

The system of functions $\{e^{t\lambda}\}_{t \in \mathbb{R}}$ is complete in the space $C([\lambda_{\min}, \lambda_{\max}])$. Therefore for each $x(\lambda) \in C([\lambda_{\min}, \lambda_{\max}])$ the limit

$$J(x) = \lim_{N \rightarrow \infty} \int_{[\lambda_{\min}, \lambda_{\max}]} x(\lambda) M_N(d\lambda) \quad (7.2)$$

exists. The mapping $J : C([\lambda_{\min}, \lambda_{\max}]) \rightarrow \mathfrak{M}_n$ is a continuous linear mapping.

Let $M(d\lambda)$ be the weak limit of the sequence of measures $M_N(d\lambda)$. The \mathfrak{M}_n -valued measure $M(d\lambda)$ gives the integral representation of the mapping J :

$$J(x) = \int_{[\lambda_{\min}, \lambda_{\max}]} x(\lambda) M(d\lambda), \quad x(\lambda) \in C([\lambda_{\min}, \lambda_{\max}]). \quad (7.3)$$

In view of (2.15),

$$J(e^{t\lambda}) = e^{tA+B}.$$

Thus the representation (7.1) is established.

If the matrix B is Hermitian: $B \in \mathfrak{H}_n$, then

$$e^{tA+B} = (e^{tA+B})^*, \quad \forall t \in \mathbb{R}.$$

Hence

$$\int_{[\lambda_{\min}, \lambda_{\min}]} e^{t\lambda} M(d\lambda) = \int_{[\lambda_{\min}, \lambda_{\min}]} e^{t\lambda} (M(d\lambda))^*, \quad \forall t \in \mathbb{R}.$$

Since the system $\{e^{t\lambda}\}_{t \in \mathbb{R}}$ is complete in the space $C([\lambda_{\min}, \lambda_{\max}])$, the measures $M(d\lambda)$ and $(M(d\lambda))^*$ must coincide. In other words, the measure $M(d\lambda)$ is \mathfrak{H}_n -valued. \square

8. The measure $M(d\lambda)$ is not necessarily non-negative.

Definition 8.1. Let $S = (s_{pq})_1^n \in \mathfrak{M}_n$. The matrix S is said to be *non-negative* if the inequality

$$\sum_{1 \leq p, q \leq n} s_{pq} \xi_p \bar{\xi}_q \geq 0$$

holds for all complex numbers ξ_1, \dots, ξ_n .

Definition 8.2. Let \mathfrak{B} be the class of Borel sets of \mathbb{R} , $M(d\lambda) : \mathfrak{B} \rightarrow \mathfrak{M}_n$ be a matrix valued measure. The measure $M(d\lambda)$ is said to be *non-negative* if the matrix $M(\delta)$ is non-negative for every set $\delta \in \mathfrak{B}$.

Comparing the equalities (1.10) and (7.1), we conclude that

$$\mu(d\lambda) = \text{trace } M(d\lambda). \quad (8.1)$$

The following question arises naturally.

Question. Let $A \in \mathfrak{H}_n$, $B \in \mathfrak{H}_n$, and $M(d\lambda)$ be the \mathfrak{H}_n -valued measure which appears in the representation (7.1) of the function e^{tA+B} . Is the measure $M(d\lambda)$ non-negative?

The following example shows that the answer to this question is negative already for $n = 2$.

Example. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (8.2)$$

The eigenvalues of the matrix $tA + B$ are

$$\lambda_1(t) = t + \sqrt{t^2 + 1}, \quad \lambda_2(t) = t - \sqrt{t^2 + 1}. \quad (8.3)$$

The spectral projectors of the matrix $tA + B$ corresponding to these eigenvalues are

$$E_1(t) = \begin{bmatrix} \frac{\sqrt{t^2+1}+t}{2\sqrt{t^2+1}} & \frac{1}{2\sqrt{t^2+1}} \\ \frac{1}{2\sqrt{t^2+1}} & \frac{\sqrt{t^2+1}-t}{2\sqrt{t^2+1}} \end{bmatrix}, \quad E_2(t) = \begin{bmatrix} \frac{\sqrt{t^2+1}-t}{2\sqrt{t^2+1}} & -\frac{1}{2\sqrt{t^2+1}} \\ -\frac{1}{2\sqrt{t^2+1}} & \frac{\sqrt{t^2+1}+t}{2\sqrt{t^2+1}} \end{bmatrix}. \quad (8.4)$$

The matrix e^{tA+B} can be calculated explicitly:

$$e^{tA+B} = e^{\lambda_1(t)} E_1(t) + e^{\lambda_2(t)} E_2(t). \quad (8.5)$$

In particular,

$$\left(\frac{d}{dt} e^{tA+B} \right) \Big|_{t=0} = D, \quad (8.6)$$

where

$$D = \begin{bmatrix} e & \frac{e - e^{-1}}{2} \\ \frac{e - e^{-1}}{2} & e^{-1} \end{bmatrix}. \quad (8.7)$$

The matrix D is not non-negative:

$$\det D = \frac{6 - e^2 - e^{-2}}{4} < 0. \quad (8.8)$$

From (7.1) it follows that

$$D = \int_{[0,2]} \lambda M(d\lambda). \quad (8.9)$$

If the measure $M(d\lambda)$ would be non-negative, then the matrix D would be non-negative.

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